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Topologically non–trivial chiral transformations: The chiral invariant elimination of the axial vector meson[†]

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Abstract

The role of chiral transformations in effective theories modeling Quantum Chromo Dynamics is reviewed. In the context of the Nambu–Jona–Lasinio model the hidden gauge and massive Yang–Mills approaches to vector mesons are demonstrated to be linked by a special chiral transformation which removes the chiral field from the scalar–pseudoscalar sector. The chirally rotated axial vector meson field ($\tilde{A}_\mu$) transforms homogeneously under flavor rotations and may thus be dropped without violating chiral symmetry. The fermion determinant for static meson field configurations is computed by summing the discretized eigenvalues of the Dirac Hamiltonian. It is discussed how the local chiral transformation loses its unitary character in a finite model space. This technical issue proves to be crucial for the construction of the soliton within the Nambu–Jona–Lasinio model when the chirally rotated axial vector field is neglected. In the background of this soliton the valence quark is strongly bound, and its eigenenergy turns out to be negative. This important physical property which is usually generated only by non–vanishing axial vector is thus carried over by the simplification $\tilde{A}_\mu = 0$.

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1. Introduction

Since a solution to Quantum Chromo Dynamics (QCD) is not yet available one has to recede on models in order to explore processes described by the strong interaction. These models are usually constructed under the requirement that the symmetries of the underlying theory, *i.e.* QCD are maintained. In this context chiral symmetry and its spontaneous breaking are of special importance. In this article we will explore a special chiral transformation when topologically non–trivial meson field configurations like solitons are involved. To begin with, let us briefly review the relevance of chiral symmetry on the one side and solitonic field configurations on the other in the context of strong interactions.

QCD can be extended from $SU(3)$ to $SU(N_C)$ where N_C denotes the number of color degrees of freedom. It was observed by 't Hooft [1] that in the limit $N_C \to \infty$ QCD is equivalent to an effective theory of weakly interacting mesons. Subsequently Witten [2] conjectured that baryons emerge as solitons of the meson fields within this effective theory. Stimulated by Witten's conjecture much interest has been devoted to the description of baryons as chiral solitons during the past decade [3, 4]. In the soliton description of baryons the chiral field and in particular its topological structure play a key role. The topological character of the chiral field especially endows the soliton with baryonic properties like baryon charge and spin [5]. This comes about via the chiral anomaly [6] which is a unique feature of all quantum field theories where fermions live in a gauge group and couple to a chiral field. The fermionic part of such a theory has the generic structure

$$\begin{aligned} Z_F[\Phi] &= \int D\Psi D\bar{\Psi}\, \exp\left[i\int d^4x\, \bar{\Psi}\,(i\not{D} - \hat{m}_0)\,\Psi\right] \\ &= \text{Det}\,(i\not{D} - \hat{m}_0) \end{aligned} \tag{1.1}$$

where $\hat{m}_0$ denotes the current quark mass matrix which will be ignored in the ongoing discussions. Furthermore

$$i\not{D} = i\not{\partial} - \Phi = i(\not{\partial} + \not{\Gamma}) - MP_R - M^\dagger P_L \tag{1.2}$$

represents the Dirac–operator of the fermions in the external Bose–field Φ. Φ in general contains vector V_μ, axial vector A_μ fields[a] as well as scalar S and pseudo-scalar fields P

$$\Gamma_\mu = V_\mu + A_\mu\gamma_5, \qquad M = S + iP = \xi_L^\dagger \Sigma \xi_R. \tag{1.3}$$

Here $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$ are the chiral projectors. Accordingly one defines left(L)– and right(R)–handed quark fields: $\Psi_{L,R} = P_{L,R}\Psi$. The chiral field U is defined via the polar decomposition of the meson fields

$$U = \xi_L^\dagger \xi_R = \exp(i\Theta). \tag{1.4}$$

The chiral anomaly arises because there is no regularization scheme which simultaneously preserves local vector and axial vector (chiral) symmetries. In renormalizable

[a]In our notation V_μ and A_μ are anti–hermitian.

theories the chiral anomaly can be calculated in closed form and is given by the Wess-Zumino action [7, 5, 8]

$$\mathcal{A} = S_{WZ} = \frac{iN_C}{240\pi^2}\int_{M^5}(UdU^\dagger)^5 \tag{1.5}$$

where the notation of alternating differential forms has been used. M^5 denotes a five dimensional manifold whose boundary is Minkowski space. Obviously the chiral anomaly is tightly related to the chiral field since the Wess-Zumino action vanishes when the chiral field disappears ($U=1$). The chiral anomaly is, however, not merely a technical artifact but has well established physical consequences. In the meson sector it gives rise to the so–called "anomalous decay processes" like *e.g.* $\pi \to 2\gamma$ and $\omega \to 3\pi$. In the soliton sector the chiral anomaly requires for $N_C = 3$ the soliton to be quantized as a fermion and endows the soliton with half integer spin and integer baryon number[b] [5, 9].

For many purposes it is convenient to perform a chiral rotation of the fermions [10, 11]

$$\tilde{\Psi} = \Omega\Psi \quad \text{with} \qquad \Omega = P_L\xi_L + P_R\xi_R, \quad i.e. \quad \tilde{\Psi}_{L,R} = \xi_{L,R}\Psi_{L,R}. \tag{1.6}$$

This transformation defines a chirally rotated Dirac-operator

$$\Psi i\not{D}\Psi = \bar{\tilde{\Psi}} i\tilde{\not{D}}\tilde{\Psi} \tag{1.7}$$

which acquires the form

$$i\tilde{\not{D}} = \Omega^\dagger i\not{D}\Omega^\dagger = i\gamma_\mu\left(\partial^\mu + \tilde{V}^\mu + \tilde{A}^\mu\gamma_5\right) - \Sigma. \tag{1.8}$$

The chiral rotation has removed the chiral field from the scalar pseudo-scalar sector of the rotated Dirac operator $i\tilde{\not{D}}$. As a consequence the vector and axial vector fields now become chirally rotated

$$\begin{aligned}\tilde{V}_\mu + \tilde{A}_\mu &= \xi_R(\partial_\mu + V_\mu + A_\mu)\xi_R^\dagger,\\ \tilde{V}_\mu - \tilde{A}_\mu &= \xi_L(\partial_\mu + V_\mu - A_\mu)\xi_L^\dagger.\end{aligned} \tag{1.9}$$

Even in the absence of vector and axial vector fields in the original Dirac operator ($V_\mu = A_\mu = 0$) the chiral rotation induces vector and axial vector fields

$$\begin{aligned}\tilde{V}_\mu(V_\mu = A_\mu = 0) = v_\mu &= \frac{1}{2}\left(\xi_R\partial_\mu\xi_R^\dagger + \xi_L\partial_\mu\xi_L^\dagger\right),\\ \tilde{A}_\mu(V_\mu = A_\mu = 0) = a_\mu &= \frac{1}{2}\left(\xi_R\partial_\mu\xi_R^\dagger - \xi_L\partial_\mu\xi_L^\dagger\right).\end{aligned} \tag{1.10}$$

For the soliton description of baryons the chiral field is usually assumed to be of the hedgehog type

$$U = \exp\left(i\Theta(r)\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\right) \tag{1.11}$$

The non–trivial topological structure of this configuration is then exhibited by the boundary conditions $\Theta(0) = -n\pi$ and $\Theta(\infty) = 0$. The chiral field thus represents a mapping

[b] For this proof it is mandatory to consider flavor SU(3).

from the compactified coordinate space (all points at spatial infinity are identified) to SU(2) flavor space, *i.e.* $S^3 \rightarrow S^3$. The associated homotopy group, $\Pi_3(S^3)$, is isomorphic to Z, the group of integer numbers. The isomorphism is given by the winding number $(\Theta(0) - \Theta(\infty))/\pi = -n$. Assuming the unitary gauge ($\xi_L^\dagger = \xi_R$) the induced vector field is of the Wu–Yang form [11]

$$v_0 = 0, \quad v_i = i v_i^a \frac{\tau^a}{2}, \quad v_i^a = \epsilon^{ika} \hat{r}_k \frac{G(r)}{r} \tag{1.12}$$

with the profile function $G(r)$ given by the chiral angle $\Theta(r)$

$$G(r) = -2\sin^2 \frac{\Theta(r)}{2}. \tag{1.13}$$

For odd n the topological non–trivial character of the chiral rotation is also reflected by a non–vanishing value of the induced vector field at $r = 0$. The induced axial vector field $a_i = i a_i^a \tau^a/2$ becomes

$$a_i^a = \hat{r}_i \hat{r}_a \left(\Theta'(r) - \frac{\sin\Theta(r)}{r} \right) + \delta_{ia} \frac{\sin\Theta(r)}{r} \tag{1.14}$$

where the prime indicates the derivative with respect to the argument.

The use of the chirally rotated fermions is advantageous since the rotated Dirac-operator does no longer contain a chiral field and its determinant is hence anomaly free. The chiral anomaly, however, has not been lost by the chiral rotation but is hidden in the integration measure over the fermion fields. In fact, as observed by Fujikawa [12], a chiral rotation of the fermion fields gives rise to a non–trivial Jacobian of the integration measure

$$D\Psi D\bar{\Psi} = J(U) D\tilde{\Psi} D\bar{\tilde{\Psi}} \tag{1.15}$$

which is precisely given by the chiral anomaly

$$J(U) = \exp\,(i\mathcal{A}) \tag{1.16}$$

Thus we have the relation

$$-i\mathrm{Tr}\log i\not{D} = -i\mathrm{Tr}\log i\tilde{\not{D}} + \mathcal{A} \tag{1.17}$$

In many cases it is convenient to work with the chirally rotated fermion fields because of the absence of the anomaly from the fermion determinant.

As already mentioned above the chiral anomaly or equivalently the non–trivial Jacobian in the fermionic integration measure arise due to the need for regularization, which introduces a finite cut–off. In the regularized theory the chiral anomaly can be evaluated in a gradient expansion. In leading order the anomaly is then given by the Wess-Zumino action (1.5). There are, however, higher order terms which are suppressed by inverse powers of the cut–off. In renormalizable theories where the cut–off goes to infinity, these higher order terms disappear and the chiral anomaly is known in closed form. In non–renormalizable effective theories, however, the cut–off of the regularization scheme has to be kept finite and acquires a physical meaning, indicating the range of validity of the

effective theory. In this case the higher order terms of the gradient expansion do not disappear but contribute to the anomaly which is then no longer available in closed form.

Furthermore, when the soliton sector of such effective mesonic theories is studied it is not sufficient to only consider the leading and sub–leading contributions from the gradient expansion and one has to perform a full non–perturbative evaluation of the fermion determinant [13]. The non–perturbative calculations have to be performed numerically [14], with the continuous space being discretized. This is the case for both, coordinate and momentum space. Also in the non–perturbative studies of the soliton sector of the effective theory the use of the chiral rotation is in many cases advantageous [11]. As noticed above for the soliton description of baryons the topological nature of the chiral field is crucial. Actually, topology is a property of continuous spaces (manifolds) and it is *a priori* not clear whether the chiral rotation with a topologically non–trivial chiral field can be represented in a finite dimensional and discretized model space used in the numerical calculations. In this respect let us recall that a single point defect in a manifold changes its topological properties already drastically.

The purpose of this paper is twofold. First we wish to present a study of the chiral rotation in non–perturbative soliton calculations where the fermion determinant has to be numerically evaluated in the background of a topologically non–trivial chiral field. It will be demonstrated how this rotation influences the choice of the boundary conditions for the eigenstates of the Dirac Hamiltonian. Second we will make use of the fact that in the chirally rotated formulation the axial vector field may be set to zero (*i.e.* $\tilde{A}_\mu = 0$) without spoiling chiral symmetry. The resulting soliton configuration is constructed and compared with various solutions in the unrotated formulation. For definiteness we will use the Nambu-Jona-Lasinio (NJL) [15] model as a microscopic fermion theory which shares all the relevant properties of chiral dynamics with QCD. Its bosonized version gives a quite satisfactory description of mesons [10] and also of baryons when the soliton picture is assumed [16].

The organization of the paper is as follows: After these introductory remarks we review the importance of the local chiral rotation (1.6) for the extraction of meson properties from the NJL action. In that section also the chirally invariant elimination of the axial vector meson is described. In section 3 we review the computation of the soliton solution to the NJL model of pseudoscalar mesons. This may straightforwardly be generalized to the inclusion of other mesons as long as the Euclidean Dirac Hamiltonian remains Hermitian. Section 4 is devoted to the study of the local chiral rotation for topologically non–trivial field configurations and its influence on the soliton solution. The computation of the soliton solution with the rotated axial vector meson field being eliminated is described in section 5. Previously it has been shown that the presence of the axial vector field is responsible for a strong binding of the valence quark and the valence quarks' eigenenergy being negative [17, 18]. We will in particular examine whether this important piece of information is maintained after eliminating $\tilde{A}_\mu$. This is not obvious because simply setting $A_\mu = 0$ leads to a positive valence quark energy for a reasonable choice of parameters [19]. A concluding discussion is given in section 6. Some technical remarks on the boundary conditions for the Dirac spinors are left as an appendix.

2. Properties of local chiral transformations

As indicated in the introduction the chirally rotated formulation of the NJL model is suited to investigate properties of (axial–) vector mesons. In the present section we

will therefore briefly review the results and illuminate the connection with the hidden gauge symmetry (HGS) and massive Yang–Mills (MYM) approaches for the description of (axial–) vector mesons. Many of the results reported in this section are taken from earlier works [10, 20, 11, 21]. Nevertheless we repeat these results here in order to put our work into perspective and have the paper self–contained.

In the chirally rotated formulation the bosonized version of the NJL model action reads

$$\mathcal{A}_{\rm NJL} = -i{\rm Tr}\log i\tilde{\not{D}} + \mathcal{A} - \frac{1}{4G_1}{\rm tr}\left(\Sigma^2 - m^2\right) - \frac{1}{4G_2}{\rm tr}\left[\left(\tilde{V}_\mu - v_\mu\right)^2 + \left(\tilde{A}_\mu - a_\mu\right)^2\right] \quad (2.1)$$

with the constituent quark mass m being the vacuum expectation value of the scalar field Σ *i.e.* $\langle\Sigma\rangle = m$. Again we have discarded terms proportional to the current quark mass matrix. The chirally transformed (axial–) vector fields are defined in eqns (1.9) and (1.10). Next we have to face the fact that the functional trace of the logarithm in (2.1) is ultra–violet divergent and thus needs regularization. This is achieved by first continuing to Euclidean space ($x_0 = -ix_4$) and then representing the real part of the Euclidean action by a parameter integral

$$\frac{1}{2}{\rm Tr}\log\left(\tilde{\not{D}}_E^\dagger\tilde{\not{D}}_E\right) \longrightarrow -\frac{1}{2}\int_{1/\Lambda^2}^\infty \frac{ds}{s}{\rm exp}\left(-s\tilde{\not{D}}_E^\dagger\tilde{\not{D}}_E\right) \quad (2.2)$$

which introduces the cut–off Λ. This substitution is an identity up to an irrelevant additive constant for $\Lambda \to \infty$. The Euclidean Dirac operator $\tilde{\not{D}}_E$ is obtained by analytical continuation of $\tilde{\not{D}}$ to Euclidean space. The prescription (2.2) is known as the proper–time regularization [22]. For the purpose of the present paper it is sufficient to only consider the normal parts of the action. We may therefore neglect the imaginary part as well as the anomaly $\mathcal{A}$. Thus the actual starting point of our considerations is represented by

$$\begin{aligned}\mathcal{A}_{\rm NJL} \quad &= \quad -\frac{1}{2}\int_{1/\Lambda^2}^\infty \frac{ds}{s}{\rm Tr}\ {\rm exp}\left(-s\tilde{\not{D}}_E^\dagger\tilde{\not{D}}_E\right) - \frac{1}{4G_1}{\rm tr}\left(\Sigma^2 - m^2\right)\\ &\qquad -\frac{1}{4G_2}{\rm tr}\left[\left(\tilde{V}_\mu - v_\mu\right)^2 + \left(\tilde{A}_\mu - a_\mu\right)^2\right]. \end{aligned} \quad (2.3)$$

In order to extract information about the properties of the (axial–) vector mesons commonly a (covariant) derivative expansion of the fermion determinant is performed. We choose to consider a covariant derivative expansion since, in contrast to an on–shell determination of the parameters [23], it preserves gauge invariance and does not lead to artificial mass terms. Furthermore, this procedure leaves the extraction of the axial–vector meson mass unique. Continuing back to Minkowski space and substituting the scalar field Σ by its vacuum expectation value yields the leading terms [20, 11]

$$\mathcal{L}_{\rm NJL} = \frac{1}{2g_V^2}{\rm tr}\ \left(\tilde{V}_{\mu\nu}^2 + \tilde{A}_{\mu\nu}^2\right) - \frac{6m^2}{g_V^2}{\rm tr}\ \tilde{A}_\mu^2 - \frac{1}{4G_2}{\rm tr}\left[\left(\tilde{V}_\mu - v_\mu\right)^2 + \left(\tilde{A}_\mu - a_\mu\right)^2\right] + \ldots \quad (2.4)$$

Here $\tilde{V}_{\mu\nu}$ and $\tilde{A}_{\mu\nu}$ denote the field strength tensors

$$\begin{aligned}\tilde{V}_{\mu\nu} &= \partial_\mu\tilde{V}_\nu - \partial_\nu\tilde{V}_\mu + [\tilde{V}_\mu, \tilde{V}_\nu] + [\tilde{A}_\mu, \tilde{A}_\nu],\\ \tilde{A}_{\mu\nu} &= \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu + [\tilde{V}_\mu, \tilde{A}_\nu] + [\tilde{A}_\mu, \tilde{V}_\nu]\end{aligned} \quad (2.5)$$

of chirally rotated vector and axial–vector fields, respectively. Obviously in our convention these fields contain the coupling constant g_V which in the proper–time regularization is given by

$$g_V = 4\pi \left[\frac{2N_C}{3}\Gamma\left(0, \left(\frac{m}{\Lambda}\right)^2\right)\right]^{-\frac{1}{2}}. \tag{2.6}$$

For the description of the pion fields we adopt the unitary gauge for the chiral fields: $\xi = \xi_L^\dagger = \xi_R$. The pions come into the game by the non–linear realization $\xi = \exp\left(i\boldsymbol{\tau}\cdot\boldsymbol{\pi}/f\right)$. Then the last term in eqn (2.4) contains the axial–vector pion mixing which is eliminated by a corresponding shift in the axial field: $\tilde{A}_\mu \rightarrow \tilde{A}'_\mu = \tilde{A}_\mu + (ig_V^2 m^2/12fG_2)\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}$. This shift obviously provides an additional kinetic term for the pions and thus effects the pion decay constant

$$f_\pi^2 = \frac{M_A^2 - M_V^2}{4M_A^2 G_2} \tag{2.7}$$

with the (axial–) vector meson masses

$$M_V^2 = \frac{g_V^2}{4G_2} \qquad \text{and} \qquad M_A^2 = M_V^2 + 6m^2. \tag{2.8}$$

This brief summary of known results has demonstrated the usefulness of the chiral rotation (1.6) especially in the context of the derivative expansion since it eliminates the derivative of the chiral field from the fermion determinant (1.8).

The Lagrangian of the hidden gauge approach can be obtained from (2.4) by the following approximation. One neglects the kinetic parts for the axial–vector field $\tilde{A}'_\mu$ which leaves this field only as an auxiliary field. This allows to employ the corresponding equation of motion to eliminate $\tilde{A}'_\mu$ resulting in

$$\mathcal{L} \sim \frac{1}{2g_V^2}\text{tr}\ \tilde{V}_{\mu\nu}^2 - af_\pi^2\ \text{tr}\ \left(\tilde{V}_\mu - v_\mu\right)^2 - \frac{1}{4}f_\pi^2\ \text{tr}\ a_\mu^2. \tag{2.9}$$

In the work of Bando et al. [24] a was left as an undetermined parameter. Here it is fixed in terms of physical quantities

$$a = \frac{M_A^2}{M_A^2 - M_V^2}. \tag{2.10}$$

Assuming the constituent quark mass $m = M_V/\sqrt{6}$ not only yields the Weinberg relation $M_A = \sqrt{2}M_V$ [25] but also the KSRF relation $a = 2$ [26].

Alternatively one might apply the same manipulations to the formulation in terms of the unrotated fields (1.3). Then the chiral field still appears in the fermion determinant and one has to deal with the covariant derivative

$$\mathcal{D}_\mu U = \partial_\mu U + [V_\mu, U] - \{A_\mu, U\}. \tag{2.11}$$

The leading terms in the Lagrangian can readily be obtained [10]

$$\mathcal{L} \sim \frac{3m^2}{2g_V^2}\text{tr}\ \left(\mathcal{D}_\mu U \mathcal{D}^\mu U^\dagger\right) + \frac{1}{2g_V^2}\text{tr}\ \left(V_{\mu\nu}^2 + A_{\mu\nu}^2\right) - \frac{1}{4G_2}\text{tr}\left(V_\mu^2 + A_\mu^2\right) \tag{2.12}$$

which exactly represent the massive Yang–Mills Lagrangian [8, 27]. Transforming the (axial–) vector fields according to (1.9) and noting that [20]

$$\mathrm{tr}\ \tilde{A}_\mu^2 = \frac{-1}{4}\mathrm{tr}\ \left(\mathcal{D}_\mu U \mathcal{D}^\mu U^\dagger\right) \tag{2.13}$$

one immediately observes that (2.12) and (2.4) describe the same physics. In the language of the NJL model the identity of the HGS and MYM approaches stems from the invariance of the module of the fermion determinant under the special chiral rotation (1.6).

After we have seen how the equivalence of the hidden gauge and massive Yang Mills Lagrangians emerge from the NJL model we next explore the transformation properties of the fields under flavor rotations $g_L,\ g_R$. These properties ploy a key role in order to demonstrate that the elimination of the axial vector meson field by setting $\tilde{A}_\mu = 0$ does not violate chiral symmetry. The flavor rotations are defined for the unrotated left– and right–handed quark fields

$$\Psi_L \rightarrow g_L \Psi_L \qquad \text{and} \qquad \Psi_R \rightarrow g_R \Psi_R. \tag{2.14}$$

The term which describes the coupling of the quarks to the scalar and pseudoscalar mesons is left invariant by demanding

$$\xi_L^\dagger \Sigma \xi_R \rightarrow g_L \xi_L^\dagger \Sigma \xi_R g_R^\dagger \tag{2.15}$$

which introduces the hidden gauge transformation h [10]

$$\xi_L \rightarrow h^\dagger \xi_L g_L^\dagger \quad , \qquad \xi_R \rightarrow h^\dagger \xi_R g_R^\dagger \qquad \text{and} \qquad \Sigma \rightarrow h^\dagger \Sigma h. \tag{2.16}$$

Obviously the scalar fields transform homogeneously under the hidden gauge transformation. In this context it is important to note that h may not be chosen independently but rather depends on the gauge adopted for the chiral fields. Consider *e.g.* the unitary gauge $\xi_L^\dagger = \xi_R = \xi$. This requires the transformation property

$$\xi \rightarrow g_L \xi h = h^\dagger \xi g_R^\dagger. \tag{2.17}$$

For vector type transformations $g_L = g_R = g_V$ this equation is obviously solved by $h = g_V^\dagger$. Contrary, for axial type transformations $g_L = g_R^\dagger = g_A$ h is obtained as the solution to $g_A \xi h = h^\dagger \xi g_A$ which depends on the field configuration ξ. Thus even for global flavor transformations $g_{A,V}$ the hidden gauge transformation h may be coordinate–dependent for coordinate dependent $\xi(x)$ [28, 29]. The unrotated (axial–) vector fields are required to transform inhomogeneously under the flavor rotations

$$V_\mu + A_\mu \rightarrow g_R \left(\partial_\mu + V_\mu + A_\mu\right) g_R^\dagger \qquad \text{and} \qquad V_\mu - A_\mu \rightarrow g_L \left(\partial_\mu + V_\mu - A_\mu\right) g_L^\dagger. \tag{2.18}$$

It is then straightforward to verify that the flavor transformation of the rotated fields only involves the hidden symmetry transformation h [10]

$$\tilde{\Psi}_{L,R} \rightarrow h^\dagger \tilde{\Psi}_{L,R} \quad , \qquad \tilde{V}_\mu \rightarrow h^\dagger \left(\partial_\mu + \tilde{V}_\mu\right) h \qquad \text{and} \qquad \tilde{A}_\mu \rightarrow h^\dagger \tilde{A}_\mu h. \tag{2.19}$$

The fact that $\tilde{A}_\mu$ transforms homogeneously has the important consequence that (as already mentioned) one can set $\tilde{A}_\mu = 0$ without breaking chiral symmetry bcause the last relation in (2.19) does not induce any inhomogeneity. Furthermore the vector mesons $\tilde{V}_\mu$ are not affected by this choice. Let us next examine the NJL model defined by $\tilde{A}_\mu = 0$ in more detail. The corresponding Dirac operator reads

$$i\tilde{D\!\!\!/} = i\gamma_\mu \left(\partial^\mu + \tilde{V}^\mu\right) - \Sigma \tag{2.20}$$

and the derivative expansion (2.4) simplifies to

$$\frac{1}{2g_V^2}\mathrm{tr}\ \tilde{V}_{\mu\nu}^2 - \frac{1}{4G_2}\mathrm{tr}\left[\left(\tilde{V}_\mu - v_\mu\right)^2 + a_\mu^2\right] + \ldots\ . \tag{2.21}$$

Identifying $\tilde{V}_\mu$ with the physical ρ–meson field determines the coupling constant G_2

$$m_\rho^2 = \frac{g_V^2}{4G_2} = \frac{2\pi^2}{G_2\Gamma\left(0, \frac{m^2}{\Lambda^2}\right)} \tag{2.22}$$

for $N_C = 3$. One may rewrite the last term in equation (2.21) in terms of the chiral field U (see also eqn. (2.13))

$$\mathrm{tr}\ a_\mu^2 = -\frac{1}{4}\mathrm{tr}\ \partial_\mu U \partial^\mu U \tag{2.23}$$

which provides an additional relation for G_2 in terms of the pion decay constant

$$f_\pi^2 = \frac{1}{4G_2}. \tag{2.24}$$

This finally implies $m_\rho^2 = 8\pi^2 f_\pi^2/\Gamma\left(0, m^2/\Lambda^2\right)$. It should be remarked that the above derived relations are also obtained when (in a derivative expansion) A_μ is set to zero and V_μ identified with the ρ–meson. This, however, is not surprising since these relations for the vector mesons stem from the part of the Lagrangian which does not contain pion fields. In the absence the pion fields $\tilde{V}_\mu$ and V_μ are identical, *cf.* eqn (1.9).

The derivative expansion (2.21) also contains the $\rho\pi\pi$ vertex

$$\frac{g_{\rho\pi\pi}}{\sqrt{2}}\boldsymbol{\rho}_\mu \cdot (\boldsymbol{\pi} \times \boldsymbol{\pi}) = \frac{1}{2G_2}\mathrm{tr}\left(\tilde{V}_\mu v^\mu\right). \tag{2.25}$$

Expanding the RHS in terms of the physical fields $\boldsymbol{\rho}_\mu$ and $\boldsymbol{\pi}$ yields

$$g_{\rho\pi\pi} = \frac{m_\rho}{\sqrt{2} f_\pi} \tag{2.26}$$

which falls short a factor $\sqrt{2}$ of the KSRF relation. This seems to indicate that some relevant information is lost by setting $\tilde{A}_\mu = 0$. Whether this is also the case in the soliton sector will be explored in section 5.

In phenomenological vector meson models frequently a term like

$$\mathrm{tr}\left[U\left(V_\mu + A_\mu\right)U^\dagger\left(V_\mu - A_\mu\right)\right] \tag{2.27}$$

is added [29, 30] to reproduce the empirical value for $g_{\rho\pi\pi}$ when setting $\tilde{A}_\mu = 0$. This term is invariant under global chiral rotations. The absence of this term in the NJL model causes the improper prediction for this coupling constant.

3. The NJL soliton

In the baryon number one sector the NJL model has the celebrated feature to possess localized static solutions with finite energy, *i.e.* solitons [13, 14]. Here we wish to briefly review this solution for the case of pseudoscalar fields.

For static field configurations it is convenient to introduce a Dirac Hamiltonian

$$\mathcal{H} = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta \left(P_R \xi \langle \Sigma \rangle \xi + P_L \xi^\dagger \langle \Sigma \rangle \xi^\dagger \right) \tag{3.1}$$

where we have assumed the unitary gauge (*i.e.* $\xi_L^\dagger = \xi_R = \xi$). This Hamiltonian enters the Euclidean Dirac operator via

$$i\beta \not{D}_E = -\partial_\tau - \mathcal{H}. \tag{3.2}$$

For static mesonic background fields the fermion determinant can conveniently be expressed in terms of the eigenvalues ϵ_μ of the Dirac Hamiltonian[a]

$$\mathcal{H}\Psi_\mu = \epsilon_\mu \Psi_\mu. \tag{3.3}$$

These eigenvalues are functionals of the mesonic background fields. Depending of the specific boundary condition (which fixes the quantum reference state) to the fermion fields in the functional integral (1.1), the fermion determinant (1.1) contains in general besides a vacuum part $\mathcal{A}^0$ also a valence quark part $\mathcal{A}^{\rm val}$ [13]

$$\mathcal{A} = \mathcal{A}^0 + \mathcal{A}^{\rm val}. \tag{3.4}$$

The valence quark part arising from the explicit occupation of the valence quark levels is given by

$$\mathcal{A}^{\rm val} = -E^{\rm val}[\xi]T \ , \quad E^{\rm val}[\xi] = N_C \sum_\mu \eta_\mu |\epsilon_\mu|. \tag{3.5}$$

Here $\eta_\mu = 0, 1$ denote the occupation numbers of the valence (anti-) quark states. These have to be adjusted such the total baryon number

$$B = \sum_\mu \left(\eta_\mu - \frac{1}{2} {\rm sgn}\, (\epsilon_\mu) \right) \tag{3.6}$$

equals unity. The vacuum part is conveniently evaluated for infinite Euclidean times ($T \to \infty$) which fixes the vacuum state as the quantum reference state (no valence quark orbit occupied). For the present considerations it will be sufficient to evaluate the real vacuum part

$$\mathcal{A}_R^0 = \frac{1}{2} {\rm Tr}\, \log \not{D}_E^\dagger \not{D}_E. \tag{3.7}$$

[a]The treatment generalizes to more complicated field configurations as long as $\mathcal{H}$ is Hermitian (*cf.* section 5).

Since for static configurations one has $[\partial_\tau, \mathcal{H}]$=0 and thus $\not{D}_E^\dagger \not{D}_E = -\partial_\tau^2 + \mathcal{H}^2$. Then it is straightforward to evaluate the real part of the fermion determinant in proper–time regularization[b] [13]

$$\begin{aligned}
\mathcal{A}_R^0 &= -\frac{1}{2}\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \mathrm{Tr}\, \exp\left(-s \not{D}_E^\dagger \not{D}_E\right) \\
&= -T\frac{N_C}{2}\int_{-\infty}^{\infty} \frac{dz}{2\pi} \sum_\mu \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left(-s\left[z^2 + \epsilon_\mu^2\right]\right). \qquad (3.8)
\end{aligned}$$

The temporal part of the trace has become the z integration. As this integral is Gaussian it can readily be carried out yielding

$$\mathcal{A}_R^0 = -T\frac{N_C}{2}\int_{1/\Lambda^2}^{\infty} \frac{ds}{\sqrt{4\pi s^3}} \sum_\mu \exp\left(-s\epsilon_\mu^2\right). \qquad (3.9)$$

This expression allows to read off the static energy functional $E[\xi]$ since $\mathcal{A}_R^0 \to -TE^0[\xi]$ as $T \to \infty$

$$E^0[\xi] = \frac{N_C}{2}\int_{1/\Lambda^2}^{\infty} \frac{ds}{\sqrt{4\pi s^3}} \sum_\mu \exp\left(-s\epsilon_\mu^2\right). \qquad (3.10)$$

Finally the total energy functional is given by

$$E[\xi] = E^{\mathrm{val}}[\xi] + E^0[\xi] - E^0[\xi = 1] \qquad (3.11)$$

which is normalized to the energy of the vacuum configuration $\xi = 1$. In the chiral limit ($m_\pi = 0$), which we have adopted here, the meson part of the action does not contribute to the soliton energy. The chiral soliton is the ξ configuration which minimizes $E[\xi]$ and the minimal $E[\xi]$ is then identified as the soliton mass.

To be specific we employ the hedgehog *ansatz* for the chiral field

$$\xi(\boldsymbol{r}) = \exp\left(\frac{i}{2}\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\ \Theta(r)\right) \qquad (3.12)$$

while the scalar fields are constrained to the chiral circle, *i.e.* $\langle\Sigma\rangle = m$. Substituting this *ansatz* into the Dirac Hamiltonian (3.1) yields

$$\mathcal{H} \;=\; \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m\left(\cos\Theta(r) + i\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\ \sin\Theta(r)\right). \qquad (3.13)$$

The stationary condition $\delta E[\xi]/\delta\xi = 0$ is made explicit by functionally differentiating the energy–eigenvalues ϵ_μ with respect to Θ

$$\frac{\delta\epsilon_\mu}{\delta\Theta(r)} = m\int d\Omega\ \Psi_\mu^\dagger(\boldsymbol{r})\beta\left(-\sin\Theta(r) + i\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\ \cos\Theta(r)\right)\Psi_\mu(\boldsymbol{r}). \qquad (3.14)$$

This leads to the equation of motion [14]

$$\cos\Theta(r)\ \mathrm{tr}\int d\Omega\ \rho_S(\boldsymbol{r},\boldsymbol{r}) i\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}} \ = \sin\Theta(r)\ \mathrm{tr}\int d\Omega\ \rho_S(\boldsymbol{r},\boldsymbol{r}) \qquad (3.15)$$

[b]The imaginary part does not contribute for the field configurations under consideration because $\mathcal{H}$ is assumed to be Hermitian.

where the traces are over flavor and Dirac indices only. According to the sum (3.11) the scalar quark density matrix $\rho_S(\boldsymbol{x},\boldsymbol{y})) = \langle q(\boldsymbol{x})\bar{q}(\boldsymbol{y})\rangle$ is decomposed into valence quark and Dirac sea parts:

$$\begin{aligned} \rho_S(\boldsymbol{x},\boldsymbol{y}) &= \rho_S^{\rm val}(\boldsymbol{x},\boldsymbol{y}) + \rho_S^{\rm vac}(\boldsymbol{x},\boldsymbol{y}) \\ \rho_S^{\rm val}(\boldsymbol{x},\boldsymbol{y}) &= \sum_\mu \Psi_\mu(\boldsymbol{x})\eta_\mu\bar{\Psi}_\mu(\boldsymbol{y}){\rm sgn}(\epsilon_\mu) \\ \rho_S^{\rm vac}(\boldsymbol{x},\boldsymbol{y}) &= \frac{-1}{2}\sum_\mu \Psi_\mu(\boldsymbol{x}){\rm erfc}\left(\left|\frac{\epsilon_\mu}{\Lambda}\right|\right)\bar{\Psi}_\mu(\boldsymbol{y}){\rm sgn}(\epsilon_\mu). \end{aligned} \tag{3.16}$$

The explicit form of the eigenfunctions $\Psi_\mu(\boldsymbol{x})$ as well as remarks on the appropriate boundary conditions may be found in the appendix.

4. The chirally rotated fermion determinant

In the appendix it is demonstrated that the normalizable solutions to the free Dirac equation with spherical boundary conditions represent a pertinent basis for the diagonalization of the Dirac Hamiltonian with the soliton present. This property is based on the fact that the Hamiltonian (3.13) is free of singularities. In this section we will explain how singularities appearing in a Dirac Hamiltonian influence the choice of basis states. Let us for this purpose consider the Hamiltonian for the chirally rotated quark fields $\tilde{\Psi} = \Omega(\Theta)\Psi$ (*cf.* eqns (1.12)–(1.14) and ref. [11]):

$$\begin{aligned} \mathcal{H}_R = \Omega(\Theta)\mathcal{H}\Omega^\dagger(\Theta) &= \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m - \frac{1}{2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}})\left(\Theta'(r) - \frac{1}{r}\sin\Theta(r)\right) \\ &\quad -\frac{1}{2r}(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})\sin\Theta(r) - \frac{1}{r}\boldsymbol{\alpha}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{\tau})\sin^2\left(\frac{\Theta(r)}{2}\right) \end{aligned} \tag{4.1}$$

since in unitary gauge $\Omega(\Theta) = \cos(\Theta/2) + i\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\,\sin(\Theta/2)$. Obviously the Θ–dependence in the Hamiltonian has been transferred to induced (axial–) vector meson fields. As expected the rotated Hamiltonian, $\mathcal{H}_R$, contains an explicit singularity in the $1/r\boldsymbol{\alpha}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{\tau})\sin^2\left(\frac{\Theta(r)}{2}\right)$ term at $r=0$. Additionally there are "coordinate singularities" in the expressions involving $\hat{\boldsymbol{r}}$. All these singularities appear because the "coordinate singularity" in $\Omega(\Theta)$ is not compensated by corresponding values of the chiral angle Θ. Stated otherwise: the transformation $\Omega(\Theta)$ with $\Theta(0) - \Theta(\infty) = -n\pi$ is topologically distinct from the unit transformation. Although $\Omega(\Theta)$ represents a unitary transformation it is then not astonishing that a numerical diagonalization

$$\mathcal{H}_R\tilde{\Psi}_\mu = \tilde{\epsilon}_\mu\Psi_\mu \tag{4.2}$$

in the basis of the free Hamiltonian does <u>not</u> render the eigenvalues of the unrotated Hamiltonian, $\mathcal{H}$, (*i.e.* $\tilde{\epsilon}_\mu \neq \epsilon_\mu$) despite the relevant matrix elements being finite. This finiteness is merely due to the r^2 factor in the volume element. One might suspect that the Hamiltonian $\mathcal{H}_{2R} = \Omega(2\Theta)\mathcal{H}\Omega^\dagger(2\Theta)$ obtained by a 2Θ rotation has the same spectrum as $\mathcal{H}$, since $\mathcal{H}_{2R}$ is free of singularities. Although this behavior is exhibited by the numerical solution for the low–lying energy eigenvalues, the topological character of the transformation has drastic consequences for the states at the lower and upper ends of the spectrum in momentum space. Adopting the same basis states for diagonalizing $\mathcal{H}$ and $\mathcal{H}_{2R}$ one observes that the eigenvalues of $\mathcal{H}_{2R}$ are shifted against those of $\mathcal{H}$,

i.e. the most negative energy eigenvalue is missing while an additional one has popped up at the upper end of the spectrum. Up to numerical uncertainties the eigenvalues in the intermediate region agree for both $\mathcal{H}$ and $\mathcal{H}_{2R}$. This behavior is sketched in figure 4.1 and repeats itself for $\mathcal{H}_{4R} = \Omega(4\Theta)\mathcal{H}\Omega^\dagger(4\Theta)$. Thus the chiral rotation represents another example of the so–called "infinite hotel story" [31] which is an interesting feature reflecting the topological character of this transformation.

Let us now return to the problem of diagonalizing $\mathcal{H}_R$ and restrict ourselves for the moment to the channel $G^\Pi = 0^+$. The grand spin operator $\boldsymbol{G}$ is defined in the appendix (A.1). It should be stressed that $[\Omega, \boldsymbol{G}] = 0$. Thus the local chiral rotation may be investigated in each grand spin channel separately.

At $r = 0$ the chiral rotation

$$\Omega(r = 0) = -i\,(\boldsymbol{\tau} \cdot \hat{\boldsymbol{r}})\,\gamma_5 \tag{4.3}$$

obviously exchanges upper and lower components of Dirac spinors[a]. The corresponding wave–functions $\tilde{\Psi}_\mu^{(0,+)}(r = 0) = \Omega(r = 0)\Psi_\mu^{(0,+)}(r = 0)$ (see eqn (A.9)) can obviously not be represented by the free basis as in general the lower component of $tilde\Psi_\mu^{(0,+)}(r = 0)$ is different from zero. However, the lower components of the eigenstates of the free Hamiltonian in the $G^\Pi = 0^+$ channel have this property (*cf.* eqns (A.3,A.5)). It is thus clear that the local chiral rotation is not unitary in a finite model space. Furthermore the equality $\epsilon_\mu = \tilde{\epsilon}_\mu$ cannot be gained without additional manipulations. Of course, this "non–unitarity" is completely due to the topological character of this rotation. This problem can be avoided by defining a basis in the topologically non–trivial sector via

$$\tilde{\Psi}_{\mu 0} = \Omega(\phi)\Psi_{\mu 0} \tag{4.4}$$

with $\Psi_{\mu 0}$ being the solutions to the free unrotated Hamiltonian. ϕ represents an auxiliary radial function satisfying the boundary conditions $\phi(0) = -\pi$ and $\phi(D) = 0$. *E.g.* we may take[b]

$$\phi(r) = -\pi\left(1 - \frac{r}{D}\right)\exp\left(-tmr\right) \tag{4.5}$$

with t being a free parameter. The diagonalization of $\mathcal{H}_R$ in the basis $\tilde{\Psi}_0$ is equivalent to diagonalizing

$$\begin{aligned}
\boldsymbol{\alpha} \cdot \boldsymbol{p} + m\beta\left(\cos\phi(r) + i\gamma_5 \boldsymbol{\tau} \cdot \hat{\boldsymbol{r}} \sin\phi(r)\right) \\
+\frac{1}{2}\left[\phi'(r) - \Theta'(r) + \frac{1}{r}\sin(\Theta(r) - \phi(r))\right]\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}\boldsymbol{\tau} \cdot \hat{\boldsymbol{r}} \\
+\frac{1}{2r}\sin(\phi(r) - \Theta(r))\boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \frac{1}{2r}\left[1 - \cos(\Theta(r) - \phi(r))\right]\boldsymbol{\alpha} \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{\tau})
\end{aligned} \tag{4.6}$$

in the standard basis $\{\Psi_{\mu 0}\}$. At this point it should be stressed again that ϕ is not a dynamical field but merely an auxiliary field which transforms the basis such as to eliminate the $1/r$–singularities from the Dirac Hamiltonian. This property is completely

[a] In the standard representation $\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

[b] D denotes the size of the spherical cavity which serves to discretize the momentum eigenstates, *cf.* the appendix.

determined by the boundary values of ϕ. Thus the results ought to be independent of the parameter t. We have confirmed this property numerically. The total energy of the soliton varies only by fractions of MeV in a wide range of t. This is, of course, negligibly small since the inherited mass scale is given by the constituent quark mass, m which is of the order of several hundred MeV. By using the locally transformed basis (4.4) also the wave–functions corresponding to the eigenstates of $\mathcal{H}_R$ agree reasonably well with the rotated wave–functions $\Omega(\Theta)\Psi_\mu$ of the original Dirac Hamiltonian $\mathcal{H}$. It should be noted that a large number of momentum states is required to numerically gain this result. This is not surprising since in order to represent the functional unity an infinite number of momentum states is needed.

At this point we want to add a remark on a further application of the chiral rotation. Using $\Theta \equiv \pm\pi$, *i.e.* a chiral transformation $\Omega = \pm i\gamma_5 \boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}$, the rotated Hamiltonian (4.1) is given by

$$\mathcal{H}_R^{\Theta=\pm\pi} = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m - \frac{1}{2}\boldsymbol{\alpha}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{\tau}) \tag{4.7}$$

whereas the unrotated one (see eqn (3.13)) is simply $\mathcal{H}^{\Theta=\pm\pi} = \boldsymbol{\alpha}\cdot\boldsymbol{p} - \beta m$. The latter is diagonalized straightforwardly. It has the same eigenvalues as the free Hamiltonian and the eigenfunctions $\Psi^{\Theta=\pm\pi}$ differ by the substitution $m \to -m$. Therefore the Hamiltonian (4.7) has also the eigenvalues of the free Hamiltonian. The eigenfunctions are given by $\Psi_R^{\Theta=\pm\pi} = \Omega^\dagger(\Theta=\pm\pi)\Psi^{\Theta=\pm\pi} = \pm i\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\Psi^{\Theta=\pm\pi}$. *I.e.* the eigenfunctions of (4.7) are easily constructed: start with the free eigenfunctions (A.3-A.5), substitute $m \to -m$ and apply $\pm i\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}$. This constitutes just one example how the chiral rotation (1.6) can be used to diagonalize non–trivial operators.

Before turning to the detailed discussion of the equation of motion in the rotated system we would like to mention that the self–consistent profile being obtained from the unrotated problem also minimizes the soliton mass in the chirally rotated frame. Stated otherwise: each change in this profile function leads to an increase of the energy obtained from the eigenvalues of $\mathcal{H}_R$.

As in the formulation within the unrotated frame the equation of motion is gained by extremizing the energy functional (3.11). The energy eigenvalues in the rotated frame, however, exhibit a different functional dependence on the chiral field. The functional derivative of these eigenvalues with respect to the chiral angle reads

$$\begin{aligned}\frac{\delta\epsilon_\mu}{\delta\Theta(r)} \;=\; & \int d\Omega\, \Big\{\frac{1}{2}\Big(\frac{\partial}{\partial r} + \frac{1}{r}\cos\Theta(r)\Big) r^2 \tilde{\Psi}_\mu^\dagger(\boldsymbol{r})\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\tilde{\Psi}_\mu(\boldsymbol{r}) \\ & -\frac{r}{2}\cos\Theta(r)\tilde{\Psi}_\mu^\dagger(\boldsymbol{r})\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\tilde{\Psi}_\mu(\boldsymbol{r}) - \frac{r}{2}\sin\Theta(r)\tilde{\Psi}_\mu^\dagger(\boldsymbol{r})\boldsymbol{\alpha}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{\tau})\tilde{\Psi}_\mu(\boldsymbol{r})\Big\}\end{aligned} \tag{4.8}$$

with $\tilde{\Psi}_\mu(\boldsymbol{r})$ being the eigenstates of the rotated Dirac Hamiltonian (4.1). The derivatives (4.8) enter the stationary condition for the energy functional resulting in the equation of motion

$$A_L(r) + A_T(r)\cos\Theta(r) - V(r)\sin\Theta(r) = 0. \tag{4.9}$$

In order to display the radial functions A_L, A_T and V it is convenient to introduce the charge density $\tilde{\rho}_C = \langle q(\boldsymbol{x})q(\boldsymbol{y})^\dagger\rangle = \tilde{\rho}_C^{\rm val} + \tilde{\rho}_C^{\rm vac}$ involving the eigenstates $\tilde{\Psi}_\mu$ of $\mathcal{H}_R$ (*cf.* eqn

(3.16)) [13]:

$$
\begin{aligned}
\tilde{\rho}_C^{\rm val}(\boldsymbol{x},\boldsymbol{y}) &= \sum_\mu \tilde{\Psi}_\mu(\boldsymbol{x})\eta_\mu\tilde{\Psi}_\mu^\dagger(\boldsymbol{y}){\rm sgn}(\epsilon_\mu) \\
\tilde{\rho}_C^{\rm vac}(\boldsymbol{x},\boldsymbol{y}) &= \frac{-1}{2}\sum_\mu \tilde{\Psi}_\mu(\boldsymbol{x}){\rm erfc}\left(\left|\frac{\epsilon_\mu}{\Lambda}\right|\right)\tilde{\Psi}_\mu^\dagger(\boldsymbol{y}){\rm sgn}(\epsilon_\mu).
\end{aligned}
\tag{4.10}
$$

The radial functions A_L, A_T and V are of axial–($A_{L,T}$) and vector(V) character

$$A_L(r) = \frac{1}{r}\frac{\partial}{\partial r}\,{\rm tr}\int d\Omega\; r^2\;\tilde{\rho}_C(\boldsymbol{r},\boldsymbol{r})\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}} \tag{4.11}$$

$$A_T(r) = {\rm tr}\int d\Omega\;\tilde{\rho}_C(\boldsymbol{r},\boldsymbol{r})\left[\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}-\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\right] \tag{4.12}$$

$$V(r) = {\rm tr}\int d\Omega\;\tilde{\rho}_C(\boldsymbol{r},\boldsymbol{r})\boldsymbol{\alpha}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{\tau}). \tag{4.13}$$

It should be noted that A_L does not depend on the auxiliary field ϕ. It is then straightforward to verify that for any meson configuration A_L and A_T satisfy the relation

$$A_L(r=0) = -(-1)^k A_T(r=0) \tag{4.14}$$

where k is defined by the value of the auxiliary field ϕ at the origin $\phi(r=0)=k\pi$. The vector type radial function V vanishes at the origin. Thus the equation of motion (4.9) together with the relation (4.14) yield the boundary condition $\Theta(r=0)=(2n+1)\pi$ for $k=1$. This is stronger than the boundary condition derived from the original equation of motion (3.15) which also allows for even multiples of π for $\Theta(r=0)$ since ${\rm tr}\int d\Omega\;\rho_S(\boldsymbol{r}=0,\boldsymbol{r}=0)i\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}=0$ Assuming the Kahana–Ripka [32] boundary conditions for the unrotated basis states $\Psi_{\mu 0}$ similar considerations for $r=D$ show that $\Theta(r=D)=2l\pi$ since $A_L(r=D)=-A_T(r=D)$ as long as $\phi(D)=0$. Obviously the topological charge associated with the chiral field in the hedgehog *ansatz* $(\Theta(r=0)-\Theta(r=D))/\pi$ can assume odd values only when k is odd. In particular $\phi\equiv 0$ is prohibited in the case of unit baryon number. Thus the study of the boundary conditions in the chirally transformed system corroborates the conclusion drawn from investigating the eigenvalues and –states of $\mathcal{H}_R$ that it is mandatory to also transform the basis spinors and in particular the boundary conditions to the topological non–trivial sector.

We would also like to remark that this kind of singularities does not only appear when the Dirac Hamiltonian is considered. Such topological defects have also caused problems in Skyrme type models when fluctuations off vector meson solitons were investigated [33]. In that case the boundary conditions for the vector meson fluctuations had to undergo a special gauge transformation which corresponds to the transformation of the basis quark spinors described here (4.4).

Before discussing the numerical treatment of eqn (4.9) we would like to make the remark that substituting the transformation $\Psi_\mu(\boldsymbol{r})=\Omega^\dagger(\Theta)\tilde{\Psi}_\mu(\boldsymbol{r})$ into the original equation of motion (3.15) does not result in the relation (4.8) but rather yields the constraint

$$0=\sum_\mu\left(\eta_\mu{\rm sgn}(\epsilon_\mu)-\frac{1}{2}{\rm erfc}\left(\left|\frac{\epsilon_\mu}{\Lambda}\right|\right)\right)\bar{\tilde{\Psi}}_\mu(\boldsymbol{r})\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\tilde{\Psi}_\mu(\boldsymbol{r}). \tag{4.15}$$

Thus the equation of motion (4.9) cannot be obtained by transforming the states Ψ_μ but only by employing the Dirac equation in the rotated frame (4.2) to extremize the energy

functional. The constraint (4.15) does not represent an over–determination of the system since infinitely many states are involved.

Due to the transcendent character of the equation of motion in the rotated frame (4.9) solutions cannot be obtained for arbitrary values of the radial functions A_L, A_T and V. *E.g.* for large distances $\Theta \to 0$ requires $|A_L| \le |A_T|$ in order to find a solution to eqn (4.9). Thus the treatment of the NJL soliton in the chirally rotated frame is not very well suited for an iterative procedure to find the self–consistent solution. The reason is that a small deviation of the radial functions A_L, A_T and V from those corresponding to this solution can render eqn (4.9) indissoluble for $\Theta(r)$. Then it is not unexpected that at large distances the solution to the rotated equation of motion (4.9) becomes unstable and the original profile function cannot be reproduced for $r \ge 2$fm. For smaller values of r the original profile function is well reproduced. In figure 4.2 this behavior is displayed. The self–consistent solution to eqn (3.15) serves as ingredient to evaluate the radial functions A_L, A_T and V. The solution to eqn (4.9) is then constructed and compared to the original profile function.

5. The soliton without axial vector mesons

In this section we will examine the soliton solutions when the chirally rotated axial vector field is set to zero, $\tilde{A}_\mu = 0$. In Skyrme type models this approach has been frequently used to study the structure of baryons [34, 35, 33]. Here will ignore the effects of the ω meson. There are two reasons for doing so. First, as this field represents an isosinglet it is not affected by the chiral rotation. The second reason is of more technical nature. Since the corresponding grand spin zero ansatz introduces a non–vanishing temporal component of a vector field the Dirac Hamiltonian in Euclidean space is no longer Hermitian. A stringent derivation of the corresponding Minkowski energy functional has unfortunately not yet been performed successfully. Several attempts for motivating such an energy functional have, however, been made [37, 38, 39, 40]. Excluding the ω–meson we have besides the hedgehog ansatz for the chiral angle (1.11) only the Wu–Yang form for the chirally rotated vector field

$$\tilde{V}_\mu = (i\omega_\mu, i\tilde{v}^a_m \frac{\tau^a}{2})_\mu \tag{5.1}$$

wherein

$$\omega_\mu = 0 \quad \text{and} \quad \tilde{v}^a_m = \frac{G(r)}{r}\epsilon_{mka}\hat{r}_k. \tag{5.2}$$

Here $G(r)$ refers to the dynamical ρ–meson field and should not be confused with the induced vector field discussed in eqn (1.13). From the special form of the Dirac operator in the chirally rotated frame (2.20) it is then obvious that the chiral field only appears in the purely mesonic part of the energy functional

$$E_{\rm m} = \frac{\pi}{G_2}\int dr \left\{ (G(r) + 1 - {\rm cos}\Theta(r))^2 + \frac{1}{2}r^2\left(\Theta'(r)\right)^2 + {\rm sin}^2\Theta(r) \right\}. \tag{5.3}$$

The boundary value $\Theta(r=0) = -\pi$ then implies that $G(r=0) = -2$ [34, 35]. The stationary condition for the chiral angle now becomes a second order non–linear differential equation

$$\frac{d^2\Theta(r)}{dr^2} = -\frac{2}{r}\frac{d\Theta(r)}{dr} + \frac{1}{r^2}{\rm sin}2\Theta(r) + \frac{2}{r^2}{\rm sin}\Theta(r)\left(G(r) + 1 - {\rm cos}\Theta(r)\right). \tag{5.4}$$

Once a profile function $G(r)$ is provided this equation can be solved analogously to Skyrme model calculations.

Undoing the chiral rotation after having set $\tilde{A}_\mu = 0$ yields a Dirac Hamiltonian which contains all possible grand spin zero operators of positive parity

$$\begin{aligned}\mathcal{H} &= \boldsymbol{\alpha}\cdot\boldsymbol{p} + m\beta\left(\cos\Theta + i\gamma_5\hat{\boldsymbol{r}}\cdot\boldsymbol{\tau}\sin\Theta\right) + \frac{1}{2}\Theta'\hat{\boldsymbol{r}}\cdot\boldsymbol{\sigma}\hat{\boldsymbol{r}}\cdot\boldsymbol{\tau} \\ &\quad + \frac{\sin\Theta}{2r}\left(G+1\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\sigma} - \hat{\boldsymbol{r}}\cdot\boldsymbol{\sigma}\hat{\boldsymbol{r}}\cdot\boldsymbol{\tau}\right) + \frac{1}{2r}\left(G\cos\Theta - 1 + \cos\Theta\right)\boldsymbol{\alpha}\cdot\left(\hat{\boldsymbol{r}}\times\boldsymbol{\tau}\right).\end{aligned} \tag{5.5}$$

In account of the above mentioned boundary values for Θ and G this operator does not contain any coordinate singularities. On the contrary, the chirally rotated Dirac Hamiltonian becomes as simple as

$$\mathcal{H}_R = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m + \frac{G(r)}{2r}\boldsymbol{\alpha}\cdot\left(\hat{\boldsymbol{r}}\times\boldsymbol{\tau}\right) \tag{5.6}$$

being, however, singular at $r = 0$. Most importantly eqn (5.6) demonstrates that the eigenvalues of $\mathcal{H}$ (or $\mathcal{H}_R$), ϵ_μ, which enter the energy functional, are indeed independent of the chiral angle Θ. Then also the contribution of the fermion determinant to the energy functional $E^{\rm det} = E^0 + E^{\rm val}$ does not dependent on the chiral angle Θ, *i.e* $E^{\rm det}[\Theta, G] = E^{\rm det}[G]$. Here $E^{\rm det}$ is understood as the sum of the eigenvalues in the form displayed in eqns (3.5,3.10). In particular an infinitesimal change of Θ leaves $E^{\rm det}$ unchanged. Thus the fermion determinant does not contribute to the equation of motion for Θ (5.4). In view of the singular character of $\mathcal{H}_R$ this operator cannot be treated using the standard basis [32, 36] but rather by employing techniques which are analogous to those developed in section 4. This corresponds to use the form (5.5) with Θ substituted by a reference profile ϕ in order to compute $E^{\rm det}[G]$. Fortunately, as the only relevant information concerns the boundary values of ϕ rather than its explicit form, we may equally well employ (5.5) with Θ being the solution of (5.4).

As the fermion determinant is a functional of G it contributes to the associated equation of motion

$$G(r) = \cos\Theta(r) - 1 - \frac{N_C}{8\pi f_\pi^2}\int d\Omega \sum_\mu \tilde{\Psi}^\dagger_\mu(\boldsymbol{r})\frac{\boldsymbol{\alpha}}{2}\cdot\left(\boldsymbol{r}\times\boldsymbol{\tau}\right)\tilde{\Psi}_\mu(\boldsymbol{r}) f'(\epsilon_\mu). \tag{5.7}$$

where ϵ_μ and $\tilde{\Psi}_\mu(\boldsymbol{r})$ are the eigenvalues and –functions of $\mathcal{H}_R$. Furthermore

$$f'(\epsilon_\mu) = \eta_\mu {\rm sgn}(\epsilon_\mu) - \frac{1}{2}{\rm sgn}(\epsilon_\mu){\rm erfc}\left(\left|\frac{\epsilon_\mu}{\Lambda}\right|\right) \tag{5.8}$$

denotes the derivative of the fermion determinant with respect to ϵ_μ.

Before discussing the emergence of self–consistent solutions we will present a computation of the fermion determinant associated with the Dirac Hamiltonian $\mathcal{H}_R$ for a given profile function $G(r)$. A physically motivated profile is given by the local approximation[a] (l.a.) to the equation of motion (5.7) (see also eqn (1.13))

$$G^{\rm l.a.}(r) = \cos\Theta_{\rm s.c.}(r) - 1 \tag{5.9}$$

[a] We adopt this notation from the analogous approximation in the Skyrme model which relates the Skyrme term to the field strength tensor of the ρ–meson.

Table 5.1. Contributions to the energy for the local approximation (5.9) as functions of the constituent quark mass m. All numbers are in MeV.

m	400	500	600
$\epsilon_{\rm val}$	-399	-500	-599
$E^{\rm det}[G^{\rm l.a.}]$	150	84	61
$E_m^{\rm l.a.}$	1221	1513	1480
$E_{\rm tot}^{\rm l.a.}$	1372	1597	1541

wherein $\Theta_{\rm s.c.}(r)$ refers to the self–consistent solution to the model with pseudoscalar fields only, which is described in section 3. In the local approximation the mesonic part of the energy is given by

$$E_{\rm m}^{\rm l.a.} = \frac{\pi}{2G_2}\int dr \left\{r^2\left(\Theta'_{\rm s.c.}(r)\right)^2 + 2{\rm sin}^2\Theta_{\rm s.c.}(r)\right\}. \tag{5.10}$$

The energy eigenvalues are obtained by diagonalizing $\mathcal{H}$ with Θ being substituted with the auxiliary field ϕ as given in eqn (4.5). For the numbers listed in table 5.1 we have confirmed that these are independent of the parameter t.

Obviously the valence quark orbit is extremely bound, its energy eigenvalue being approximately $-m^b$. Numerically we observe that the upper component of the corresponding wave–function vanishes. Thus the valence quark orbit carries all features of an antiquark state. Nevertheless, the polarization of the Dirac sea only gives a minor contribution to the energy. This can be understood by noticing that a shift of the valence quark level from $\epsilon_{\rm val} \approx m$ to $\epsilon_{\rm val} \approx -m$ does not change the vacuum part of the energu where all energy levels enter with their module, see eqn (3.10). Some caution has to be taken when considering the local approximation. It may be far off the actual solution because no local minimum of $E_m^{\rm l.a.}$ exists. Applying Derek's theorem the chiral angle can be shown to collapse when G is equated to its local approximation. Nevertheless from the small contribution of the Dirac sea to the total energy we deduce that the fermion determinant only provides a small repulsive force. The local approximation furthermore suggests that this repulsive force decreases with increasing constituent quark mass m. This force should cause G to deviate from the local approximation according to the equation of motion (5.7). This deviation should in turn stablize $E_{\rm m}$ yielding a solution to the differential equation (5.4). Stated otherwise: A significant deviation from the local approximation is needed to obtain stable self–consistent solutions.

These features are actually reflected by the self–consistent solutions to eqns (5.4,5.7). For the specific case of $m = 400$MeV the profile functions are plotted in figure 5.1. Technically we have produced this solution by diagonalizing the unrotated Dirac Hamiltonian (5.5). The resulting eigenfunctions have been transformed by $\Omega(\Theta)$ and substituted into the equation of motion for G (5.7). We have then verified that the results are indeed independent of Θ. *I.e.* the calculation of the fermion determinant has been repeated using $\Theta + \delta\Theta$ without altering the results. $\delta\Theta$ has been chosen to satisfy the appropriate boundary conditions. The polarization of the Dirac sea indeed causes a sufficient deviation of G from its local approximation to yield stable solutions. On the other hand only

[b] Here it should be noted that, as a consequence of discretizing the momentum space, the smallest module of an energy eigenvalue for a free quark is $\sqrt{M^2 + (\pi/D)^2}$. The eigenvalues of free quarks are discussed in the appendix.

Table 5.2. Contributions to the energy for self–consistent solution as functions of the constituent quark mass m. All numbers are in MeV.

m	400	500	600
$\epsilon_{\rm val}$	-354	-436	-531
$E^{\rm det}[G]$	242	170	131
E_m	165	189	157
$E_{\rm tot}$	407	359	288

a small repulsive effect is obtained resulting in profile functions which have a very small spatial extension. At $r \approx 1/4$fm the chiral angle has already dropped to 1/10 of its value at $r = 0$. In the same way the total collapse is avoided, the upper component of the valence quark wave–function becomes non–vanishing. Simultaneously the corresponding energy eigenvalue increases from $-m$ as shown in table 5.2. *I.e.* particle characteristics are admixed. Furthermore table 5.2 shows that the part of the energy which is due to the fermion determinant is larger than for the local approximation (*cf.* table 5.1). This suggests that this part of the energy does not vanish for field configurations where the profiles are collapsing to a δ–like shape. In this way the total collapse is avoided and stable solutions do exist. Nevertheless the valence quark orbit remains strongly bound and possesses properties commonly attached to presence of the a_1 meson. Thus we conclude that indeed the chiral invariant elimination of the a_1 meson carries over information which would be lost if this field were simply set to zero. A further feature which can be asserted to the presence of the a_1 meson is represented by the small extension of the soliton profiles (see figure 5.1). Previously it has been demonstrated that a non–vanishing A_μ–field provides a squeeze of the chiral angle. This can *e.g.* be inferred from figure 1 of ref.[39]. Unfortunately the total energy gets very small ($\sim$ 400MeV). Hence one has to wonder whether this model can successfully applied to the description of baryons without amendments. It is obvious that a strong repulsion is called for.

From the phenomenology of the nucleon–nucleon interaction is seems likely that the inclusion of the isoscalar vector meson ω may provide (at least some) repulsion. However, as already remarked, for the non–perturbative treatment of the NJL soliton a stringent derivation of the associated Minkowski space energy functional is not available. Fortunately the current approach allows one to incorporate an approximation to the full treatment of the ω–meson. Although this approximation may be somewhat crude it should at least provide a reliable answer to the question: Does the valence quark energy remain negative?

The approximation we are going to consider relies on a power expansion of the NJL action in the ω–field[c]. The leading order is just the coupling to the baryon current $N_C \omega_\mu B^\mu$. The crudity of the approximation consists of ignoring all other terms in the expansion of the fermion determinant and assuming the leading order gradient expression for B^μ. The latter then becomes the topological current

$$B_\mu(U) = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}{\rm tr}\left\{\left(U^\dagger\partial^\nu U\right)\left(U^\dagger\partial^\rho U\right)\left(U^\dagger\partial^\sigma U\right)\right\}. \tag{5.11}$$

The mesonic part of the action, $\mathcal{A}_m$, contains a term quadratic in ω. To this end the

[c] By construction, this is an analytical functional of ω. Thus the energy is well defined in both Euclidean and Minkowski spaces.

ω–dependent parts of the Lagrangian collect up to

$$\mathcal{L}_\omega = \frac{1}{2G_2}\omega_\mu\omega^\mu + N_C\omega_\mu B^\mu(U). \tag{5.12}$$

This expression does not contain any derivative of the ω–field. Hence it may be eliminated by its stationary condition yielding the ordinary $6^{\rm th}$ order (in gradients of the chiral field) term

$$\mathcal{L}_6 = \frac{-1}{2}\epsilon_6^2 B_\mu(U)B^\mu(U) \quad \text{with} \quad \epsilon_6^2 = \frac{6\pi^2 N_C}{m_\rho^2\Gamma\left(0, m^2/\Lambda^2\right)}. \tag{5.13}$$

Adding $\int d^4x\mathcal{L}_6$ to the mesonic part of the action gives for the corresponding part of the static energy

$$E_{\rm m} = (5.3) + \frac{\epsilon_6^2}{2\pi^3}\int \frac{dr}{r^2}\left(\frac{d\Theta(r)}{dr}\right)^2 \sin^4\Theta(r). \tag{5.14}$$

In this case a stable minimum of $E_{\rm m}$ exists even if the local approximation were assumed for $G(r)$. The equation of motion for the chiral angle now becomes

$$\begin{aligned}\left[1+\frac{\epsilon_6^2}{4\pi^4 f_\pi^2}\left(\frac{\sin\Theta}{r}\right)^4\right]\frac{d^2\Theta}{dr^2} \quad &= \quad -\frac{2}{r}\frac{d\Theta}{dr} + \frac{1}{r^2}\sin2\Theta + \frac{2}{r}\sin\Theta\left(G+1-\cos\Theta\right) \\ &\quad +\frac{\epsilon_6^2}{2\pi^4 f_\pi^2}\frac{d\Theta}{dr}\frac{\sin^3\Theta}{r^4}\left(\frac{\sin\Theta}{r} - \frac{d\Theta}{dr}\cos\Theta\right).\end{aligned} \tag{5.15}$$

The solution to this equation together with (5.7) is plotted in figure 5.2. As expected the chiral angle receives a sizable extension. This is also reflected by a large mesonic part of the energy $E_m = 1607$MeV for $m = 400$MeV. In the same manner the ρ–meson profile gets wider and deviates only slightly from its local approximation. As the profiles get extended the valence quark energy tends toward $-m$ and regains its antiparticle character which was already observed in the local approximation. We conjecture that this property is common to all models which permit a sizable extension of $G(r)$. Also the contribution of the fermion determinant to the total energy decreases. *E.g.* for $m = 400$MeV we find $E^{\rm det} = 132$MeV. Together with E_m this adds up to $E_{\rm tot} = 1739$MeV.

Let us finally compare the results obtained in the present model (where the axial vector degree of freedom is eliminated in a chirally invariant way) with other treatments of the isovector (axial) vector mesons in the context of the NJL model. The numbers for the associated energies are displayed in table 5.3.

It is obvious that either setting the original axial vector degree of freedom to zero ($A_\mu = 0$) or the chirally rotated one ($\tilde{A}_\mu = 0$) amounts to totally different approaches to the soliton sector of the NJL model. While the first one yields a positive valence quark energy (for a reasonable choice of parameters), the orbit is much more strongly bound in the second approach. This certainly represents an effect due to the axial vector field $A_\mu \neq 0$. Unfortunately the total energy of the chirally rotated treatment cannot be assigned to any of the other treatments due to its smallness.

6. Conclusions

Table 5.3. Contributions to the energy for self–consistent solution in various treatments of the NJL model. Those meson fields which are allowed to be space dependent are indicated. The constituent quark mass m =400MeV is common. All numbers are in MeV.

	$\pi - \rho$ [19]	$\pi - \rho - a_1$ [17]	This model
$\epsilon_{\rm val}$	313	-222	-351
$E^{\rm det}$	711	543	240
E_m	149	393	175
$E_{\rm tot}$	861	937	415

We have investigated the role of chiral transformations for the evaluation of fermion determinants. When these transformations are topologically trivial they provide a useful tool to evaluate the chiral anomaly. Furthermore they can be used to demonstrate the equivalence between the hidden gauge and massive Yang–Mills approaches to vector mesons. As a further application of the chiral rotation we have shown that the axial vector degree of freedom can be eliminated without violating chiral symmetry. A generalization to the case when chiral fields have a topological charge different from zero is not straightforward. Even though the special transformation we have been considering is unitary its topological character prevents the eigenvalues and –vectors of the original Dirac Hamiltonian to be regained from the rotated Hamiltonian unless the boundary conditions are transformed to the topologically non–trivial sector accordingly. Furthermore we have observed that the stationary conditions to the static energy functional in the topologically distinct sectors are <u>not</u> related by the transformation of the equation of motion. The boundary conditions for the chiral field obtained from the stationary condition have been found being invariant under the chiral rotation only when the basis quark fields are taken from the topological sector associated with the chiral transformation. Diagonalization of the rotated Dirac Hamiltonian in this basis can be reformulated into a problem where the induced vector fields (1.10) belong to the topologically trivial sector (*cf.* eqn (4.6)). In order to diagonalize the resulting operator (4.6) the standard basis [32, 36] may be employed. These techniques have been applied to the case when the chirally rotated axial vector degree of freedom is absent. We have seen that in this case soliton solutions do exist and that these are totally different from those which are obtained in the NJL model with the original axial vector field being neglected. In particular, the chirally rotated vector meson contains an important information inherited from the original axial vector meson: A negative valence quark energy. Unfortunately, the chirally rotated approach as it stands cannot be considered to be realistic. This is due to the instability of the mesonic part of the energy in the local approximation for the vector meson field. We have, however, made plausible that the incorporation of the ω–meson will provide an approach suitable to describe the physics of baryons.

Furthermore we have seen that for field configurations wherein the rotated vector fields possess a reasonable spatial extension leads to a valence quark energy being of the amount $-m$. *I.e.* the valence quark has (almost) joined the negative Dirac sea; a picture which underlies Skyrme type models. Thus those amendments of the Skyrme model which drop the chirally rotated axial vector meson field [34, 35, 33] gain strong support from the investigations in a microscopic quark model presented in this paper.

Let us finally point to a possible application of the techniques developed in section 4 and 5 for a different area of physics: The quark spectrum in the background field of an

instanton field configuration. This subject has gained some interest recently for the study of the partition function of QCD[42]. Assuming temporal gauge, $A_0 = 0$, the evaluation of this spectrum can be reduced to an eigenvalue problem for the Dirac Hamiltonian

$$\boldsymbol{\alpha} \cdot \boldsymbol{p} - \boldsymbol{\alpha} \cdot \boldsymbol{A}(\boldsymbol{x}, x_4) + \beta m_0 \tag{6.1}$$

with the time coordinate x_4 acting as a parameter. Here $A_i = V^\dagger \partial_i V$ is a pure but singular gauge configuration. V may be chosen in hedgehog form relating color to coordinate space. An explicit expression is *e.g.* given in eqn (16.50) of ref.[43]. The similarity between the one–particle operators (6.1) and (5.6) is apparent[a]. In order to diagonalize (6.1) the singularity carried by $\boldsymbol{A}$ has to be removed by a topologically non–trivial transformation. Since this transformation is different for various time slices a shifting of the quark levels as shown in figure 4.1 may occur along the path $-\infty < x_4 < +\infty$.

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Appendix: Diagonlization of the Dirac Hamiltonian

Technically the discretized eigenvalues ϵ_μ of the Dirac Hamiltonian $\mathcal{H}$ (3.1,3.3,5.5) are obtained by restricting the space R_3 to a spherical cavity of radius D and demanding certain boundary conditions at $r = D$. Eventually the continuum limit $D \to \infty$ has to be considered. In order to discuss pertinent boundary conditions it is necessary to describe the structure of the eigenstates of $\mathcal{H}$. Due to the special form of the hedgehog *ansatz* the Dirac Hamiltonian commutes with the grand spin operator

$$\mathbf{G} = \mathbf{J} + \frac{\boldsymbol{\tau}}{2} = \boldsymbol{l} + \frac{\boldsymbol{\sigma}}{2} + \frac{\boldsymbol{\tau}}{2} \tag{A.1}$$

where $\mathbf{J}$ labels the total spin and $\boldsymbol{l}$ the orbital angular momentum. $\boldsymbol{\tau}/2$ and $\boldsymbol{\sigma}/2$ denote the isospin and spin operators, respectively. The eigenstates of $\mathcal{H}$ are then as well eigenstates of $\mathbf{G}$. The latter are constructed by first coupling spin and orbital angular momentum to the total spin which is subsequently coupled with the isospin to the grand spin [32]. The resulting states are denoted by $|ljGM\rangle$ with M being the projection of $\mathbf{G}$. These states obey the selection rules

$$j = \begin{cases} G + 1/2, & l = \begin{cases} G+1 \\ G \end{cases} \\ \\ G - 1/2, & l = \begin{cases} G \\ G-1 \end{cases} \end{cases} . \tag{A.2}$$

The Dirac Hamiltonian furthermore commutes with the parity operator. Thus the eigenstates of $\mathcal{H}$ with different parity and/or grand spin decouple. The coordinate space

[a] Note, however, that $\boldsymbol{A}$ contains parity violating pieces.

representation of the eigenstates $|\mu\rangle$ is finally given by

$$\Psi_\mu^{(G,+)} = \begin{pmatrix} ig_\mu^{(G,+;1)}(r)|GG+\frac{1}{2}GM\rangle \\ f_\mu^{(G,+;1)}(r)|G+1G+\frac{1}{2}GM\rangle \end{pmatrix} + \begin{pmatrix} ig_\mu^{(G,+;2)}(r)|GG-\frac{1}{2}GM\rangle \\ -f_\mu^{(G,+;2)}(r)|G-1G-\frac{1}{2}GM\rangle \end{pmatrix} \tag{A.3}$$

$$\Psi_\mu^{(G,-)} = \begin{pmatrix} ig_\mu^{(G,-;1)}(r)|G+1G+\frac{1}{2}GM\rangle \\ -f_\mu^{(G,-;1)}(r)|GG+\frac{1}{2}GM\rangle \end{pmatrix} + \begin{pmatrix} ig_\mu^{(G,-;2)}(r)|G-1G-\frac{1}{2}GM\rangle \\ f_\mu^{(G,-;2)}(r)|GG-\frac{1}{2}GM\rangle \end{pmatrix}. \tag{A.4}$$

The second superscript labels the intrinsic parity $\Pi_{\rm intr}$ which enters the parity eigenvalue via $\Pi = (-1)^G \times \Pi_{\rm intr}$. In the absence of the soliton (*i.e.* $\Theta = 0$) the radial functions $g_\mu^{(G,+;1)}(r)$, $f_\mu^{(G,+;1)}(r)$, etc. are given by spherical Bessel functions. *E.g.*

$$g_\mu^{(G,+;1)}(r) = N_k\sqrt{1+m/E}\; j_G(kr), \quad f_\mu^{(G,+;1)}(r) = N_k {\rm sgn}(E)\sqrt{1-m/E}\; j_{G+1}(kr) \tag{A.5}$$

and all other radial functions vanishing represents a solution to $\mathcal{H}(\Theta=0)$ with the energy eigenvalues $E = \pm\sqrt{k^2+m^2}$ and parity $(-1)^G$. N_k is a normalization constant.

Two distinct sets of boundary conditions have been considered in the literature. Originally Kahana and Ripka [32] proposed to discretize the momenta by enforcing those components of the Dirac spinors to vanish at the boundary which possess identical grand spin and orbital angular momentum, *i.e.*

$$g_\mu^{(G,+;1)}(D) = g_\mu^{(G,+;2)}(D) = f_\mu^{(G,-;1)}(D) = f_\mu^{(G,-;2)}(D) = 0. \tag{A.6}$$

This boundary condition has the advantage that for a given grand spin channel G only one set of basis momenta $\{k_{nG}\}$ is involved. These k_{nG} make the $G^{\rm th}$ Bessel function vanish at the boundary ($j_G(k_{nG}D)=0$). However, this boundary condition has (among others) the disadvantage that the matrix elements of flavor generators, like τ_3 are not diagonal in momentum space. If the matrix elements of the flavor generators are not diagonal in the momenta a finite moment of inertia will result even in the absence of a chiral field [13]. Stated otherwise, in this case the boundary conditions violate the flavor symmetry. This problem can be cured [36] by changing the boundary conditions for the states with $\Pi_{\rm intr} = -1$

$$g_\mu^{(G,+;1)}(D) = g_\mu^{(G,+;2)}(D) = g_\mu^{(G,-;1)}(D) = g_\mu^{(G,-;2)}(D) = 0 \tag{A.7}$$

i.e. the upper components of the Dirac spinors always vanish at the boundary. The diagonalization of the Dirac Hamiltonian (3.1) with the condition (A.7) is technically less feasible since it involves three sets of basis momenta $\{k_{nG-1}\}, \{k_{nG+1}\}$ and $\{k_{nG}\}$ for a given grand spin channel. In table A.1 we compare some properties of the two boundary conditions (A.6) and (A.7) in the case when no soliton is present. The first four quantities appearing in that table show up in various equations of motion when *e.g.* also (axial–) vector mesons are included [38, 39]. In case such a quantity is non–zero the vacuum gives a spurious contribution to the associated equation of motion. For an iterative solution to the equations of motion this spurious contribution has to be subtracted [41]. It should be noted, however, that the relations listed in table A.1 are all satisfied for both boundary conditions in the continuum limit $D \to \infty$.

So far the discussion of the boundary conditions has only effected the point $r = D$. In the context of the local chiral rotation it is equally important to consider the wave–functions at $r = 0$. As already mentioned the solutions to the Dirac equation (3.13) are

Table A.1. Properties of the two boundary conditions (A.6) and (A.7) in the baryon number zero sector. $f(r)$ represents an arbitrary radial function.

Quantity	Condition (A.6)	Condition (A.7)
$\sum_\mu \Psi_\mu^\dagger \beta\gamma_5 i\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\Psi_\mu \mathrm{erfc}\left(\left\|\frac{\epsilon_\mu}{\Lambda}\right\|\right)\mathrm{sgn}(\epsilon_\mu)=0$	yes	yes
$\sum_\mu \Psi_\mu^\dagger \boldsymbol{\alpha}\cdot(\boldsymbol{\tau}\times\hat{\boldsymbol{r}})\,\Psi_\mu \mathrm{erfc}\left(\left\|\frac{\epsilon_\mu}{\Lambda}\right\|\right)\mathrm{sgn}(\epsilon_\mu)=0$	no	yes
$\sum_\mu \Psi_\mu^\dagger \boldsymbol{\alpha}\cdot\boldsymbol{\tau}\Psi_\mu \mathrm{erfc}\left(\left\|\frac{\epsilon_\mu}{\Lambda}\right\|\right)\mathrm{sgn}(\epsilon_\mu)=0$	yes	yes
$\sum_\mu \Psi_\mu^\dagger \boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}}\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\Psi_\mu \mathrm{erfc}\left(\left\|\frac{\epsilon_\mu}{\Lambda}\right\|\right)\mathrm{sgn}(\epsilon_\mu)=0$	yes	yes
$\mathrm{tr}\left(\beta f(r)\right)=0$	yes	no
$\mathrm{tr}\left(\boldsymbol{\gamma}\cdot\boldsymbol{\tau}f(r)\right)=0$	yes	yes
$\langle\mu\|\tau_i\|\nu\rangle=0$ for $k_\mu\neq k_\nu$	no	yes

given by spherical Bessel functions in the free case, $\Theta=0$. Except of j_0 these vanish at the origin. In the case $\Theta\neq 0$ we adopt the boundary conditions $\Theta(0)=-n\pi$ thus no singularity appears in the Dirac Hamiltonian (3.13) at $r=0$. Therefore the radial parts of the quark wave–functions may be expressed as linear combinations of the solutions to the free Dirac Hamiltonian. *E.g.*

$$g_\mu^{(G,+;1)}(r)=\sum_k V_{\mu k}[\Theta]N_k\sqrt{1+m/E_{kG}}\;j_G(k_{kG}r),$$
$$f_\mu^{(G,+;1)}(r)=\sum_k V_{\mu k}[\Theta]N_k\mathrm{sgn}(E_{kG})\sqrt{1-m/E_{kG}}\;j_{G+1}(k_{kG}r) \tag{A.8}$$

where the eigenvectors $V_{\mu k}[\Theta]$ are obtained by diagonalizing the Dirac Hamiltonian in the free basis. It should be stressed that the use of the free basis is only applicable because the point singularity hidden in $\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}$ has disappeared. If singularities show up for certain field configurations the basis for diagonalizing $\mathcal{H}$ has to be altered. This has been the central issue of section 4. There we have described that the eigenvalues and –vectors can be obtained by adopting a basis which is related by a local chiral transform to the one described in eqn (A.5) together with the appropriate boundary conditions (*cf.* eqn (A.6) or (A.7)). Here we will construct an alternative basis for the $G=0$ channel by explicitly transforming the boundary conditions. This has the advantage that there is no need to introduce the auxiliary field ϕ as in eqns (4.4 and 4.5). We therefore consider the application of the rotation (4.3) at $r=0$ to the eigenstates. In the 0^+ channel this leads to

$$\Omega(r=0)\Psi_\mu^{(0,+)}(r=0)=\begin{pmatrix}-if_\mu^{(0,+;1)}(r=0)|0\tfrac{1}{2}00\rangle\\ g_\mu^{(0,+;1)}(r=0)|1\tfrac{1}{2}00\rangle\end{pmatrix}. \tag{A.9}$$

It is then obvious that a pertinent basis is given by

$$N_k\begin{pmatrix}-i\;\mathrm{sgn}(E)\sqrt{1-E/m}\;j_1(kr)|0\tfrac{1}{2}00\rangle\\ \sqrt{1+E/m}\;j_0(kr)|1\tfrac{1}{2}00\rangle\end{pmatrix}. \tag{A.10}$$

This is, of course, no longer a solution to the free Dirac equation. Moreover, a single component of this spinor does not even solve the free Klein Gordan equation. At $r=D$

the chiral rotation equals unity. Thus we demand the discretization condition $j_1(kD)=0$ according to eqn (A.6) *i.e.* the momenta are taken from the set $\{k_{n1}\}$. With this basis we have succeeded in eliminating the singularities at $r=0$ and keeping track of the boundary conditions at $r=D$. We are thus enabled to numerically diagonalize $\mathcal{H}_R$. When comparing with the eigenvalues of $\mathcal{H}$ we again encounter a form of the "infinite hotel story": one state is missing in the negative part of the spectrum while an additional shows up in the positive part. The missing state turns out to be the one at the upper end of the negative Dirac sea, *i.e.* $E\approx -m$. Then it is important to note that in addition to the states with finite k, the basis (A.10) together with the boundary condition $j_1(kD)=0$ also allows for the "constant state" with $k=0$. In the continuum limit $(D\to\infty)$ this state is absent. Including, however, this state for finite D in the process of diagonalizing $\mathcal{H}_R$ finally renders the missing state. This is not surprising since application of the inverse chiral rotation $\Omega^\dagger(r=0)$ onto this "constant state" leads to an eigenstate of the free Dirac Hamiltonian with the eigenvalue $-m$. It should be remarked that for the free unrotated problem a "constant state" with eigenvalue $-m$ is only allowed in the $G^\Pi=1^-$ channel. Although $\Omega(\Theta)$ does commute with the grand spin operator its topological character mixes various grand spin channels via the boundary conditions.

Accordingly the diagonalization of $\mathcal{H}_R$ in the $G^\Pi=0^-$ channel demands the basis states

$$N_k\begin{pmatrix} i\ \mathrm{sgn}(E)\sqrt{1-E/m}\ j_0(kr)|1\tfrac{1}{2}00\rangle \\ \sqrt{1+E/m}\ j_1(kr)|0\tfrac{1}{2}00\rangle\end{pmatrix} \tag{A.11}$$

with the boundary condition $j_1(kD)=0$ in order to be compatible with the Kahana–Ripka [32] diagonalization of $\mathcal{H}$. The additional "constant state" needed here corresponds to a state with eigenvalue $+m$ of the free unrotated Hamiltonian[a].

We have finally been able to diagonalize the chirally rotated Hamiltonian $\mathcal{H}_R$ in the $G=0$ sector by very tricky means. It should also be kept in mind that there are now additional states at the upper (from 0^+) and lower (from 0^-) ends of the spectrum which do not possess "counterstates" in the $G=0$ part of the spectrum in the unrotated problem. For the dynamics of the problem they are of no importance because their contribution to physical quantities is damped by the regularization. However, their existence reflects the topological character of the chiral rotation.

In the other channels *i.e.* $G\geq 1$ we have unfortunately not been able to construct a set of basis states which rendered the eigenvalues of the unrotated Hamiltonian along the approach described above. In the $G=0$ sector we have already seen that a "global rotation" $-i\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\gamma_5$ is needed for the basis states in order to accommodate the boundary conditions at $r=0$. Furthermore a mixture of different grand spin channels appears via the boundary condition since this "global rotation" deviates from unity at $r=D$.

[a]The additional "constant states" thus do not alter the trace of the Hamiltonian.

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FIGURE CAPTIONS

Fig. 4.1
A schematic plot of the spectrum of the rotated Hamiltonian $\mathcal{H}_{nR} = \Omega(n\Theta)\mathcal{H}\Omega^\dagger(n\Theta)$.
Fig. 4.2
Comparison of the self–consistent profile in the unrotated formulation (dashed line) and the solution to eqn (4.9) (solid line).
Fig. 5.1
The meson profiles (left) and valence quark wave–function (right) of the self–consistent solution to eqns (5.4,5.7). Also shown is the local approximation to the vector meson field $G^{\rm l.a.} = \cos\Theta - 1$. Here the constituent quark mass m =400MeV is assumed.
Fig. 5.2
Same as figure 5.1 with the $6^{\rm th}$–order term (5.14) included. The upper component of the valence quark wave–function is negligibly small.